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Nongeometry, duality twists, and the worldsheet

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Abstract: In this paper, we use orbifold methods to construct nongeometric backgrounds, and argue that they correspond to the spacetimes discussed in [1, 2]. More precisely, we make explicit through several examples the connection between interpolating orbifolds and spacetime duality twists. We argue that generic nongeometric backgrounds arising from duality twists will not have simple orbifold constructions and then proceed to construct several examples which do have consistent worldsheet descriptions.

Keywords: Global Symmetries, Superstring Vacua, String Duality, Space-Time Symmetries.

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Contents

1. Introduction

Recently there has been a lot of work in which symmetries and dualities have been exploited in order to construct string theory backgrounds [1 − 6]. In particular, these backgrounds may be described as fibrations in which the fibers undergo non-trivial monodromies around various points or cycles in the base. It has been argued that such a description is quite generic [7]. The basic idea dates back to the work of Scherk-Schwarz [8] who, in the context of classical supergravity, imposed nontrivial boundary conditions to give masses to lower dimensional fields. Adopting these ideas to string theory brought two changes. First, the global symmetries of the low energy effective action which allow one to impose nontrivial boundary conditions are broken to discrete subgroups. Secondly the (now discrete) symmetry group may be enlarged to include the full stringy duality group. In these cases the fibers are said to undergo "duality twists" [1]. Related ideas have been around for some time. In [9] for example, backgrounds are constructed by allowing for monodromies coming in the U-duality group. There are many reasons to be interested in these "twisted" compactifications. For starters, it is often the case that moduli are projected out by the boundary conditions. Moreover, they are closely related to compactifications with H-flux, supersymmetric non-Kähler vacua, and nongeometric backgrounds [5, 10 − 17].

Nongeometric backgrounds are of particular interest since relatively little is known about them. Moreover, this lack of knowledge touches on some of the key questions facing any theory of quantum gravity. What new structures will exist that were not present in the classical point particle theory? What sort of framework will be needed in order to naturally accommodate such structures? There has been progress recently addressing the larger question of how to naturally accommodate some of these structures: the development of "Generalized Complex Geometry" [18, 19] naturally incorporates the B-field, and Hull's "doubled formalism" [20], which is closely related to the "correspondence spaces" of Bouwknegt et al. [21, 22], gives nongeometric spaces a (higher dimensional) geometric description.

This paper will address the first question by focusing on the worldsheet construction of a particular class of nongeometric backgrounds. In particular we look at backgrounds which come from torus fibrations over a torus base where the torus fiber undergoes a monodromy lying in the perturbative duality group $O(d,d;\mathbb{Z})$. Following [2] we'll refer to these as monodrofolds, although they have also been called T-folds in the literature [20].[1] In [2] it was shown from the spacetime perspective that the equations of motion forced many of the fields to be fixed under the action of the imposed monodromy, projecting out moduli which would arise in a traditional compactifications. Similar conclusions were reached in [1]. Although finding the spacetime description for these backgrounds is somewhat straightforward, the worldsheet construction is often a difficult task. In fact, due to their asymmetric nature, quantum consistency is not guaranteed.

As suggested in [1, 2], duality twists correspond to interpolating orbifolds. An interpolating orbifold is just an orbifold where the group action integrates a translation along one coordinate with nontrivial transformations of the remaining coordinates or fermions. For example, one might consider orbifolding a boson X by a reflection while orbifolding a second boson Y by a $2\pi R$ shift,

$$(X \to -X\,, \quad Y \to Y + 2\pi R). \tag{1.1}$$

This should be distinguished from the product orbifold[2] which would contain 3 distinct actions,

$$(X \to -X, Y \to Y); \quad (X \to X, Y \to Y + 2\pi R); \quad (X \to -X, Y \to Y + 2\pi R)\,. \tag{1.2}$$

As one might expect, orbifolding by (1.1) corresponds in spacetime to a circle fibration in which the fiber undergoes a reflection as it encircles the base. The name "interpolating orbifold" comes from the fact that as $R \to \infty$ the orbifold returns to the unorbifolded theory, while for $R \to 0$ the theory is indistinguishable from the product orbifold in the same limit. Such orbifolds can then be used to find theories which, for example, interpolate between backgrounds with different amounts of supersymmetry [23–25].

[1] We take the term "T-fold" to be specific to monodromies coming from the T-duality group, where the monodromies in a "monodrofold" may be more general. This being said, throughout this paper the terms will be interchangeable.

[2] The product orbifold corresponds to separately compactifying Y and orbifolding by a reflection in X.

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A shift orbifold alone is modular covariant.[3] Namely, defining ${\mathcal{Z}_R}^a{}_b$ to be the contribution to the shift partition function coming from a twists and b insertions,

$$\mathcal{Z}_R(\tau+1)^a{}_b = \mathcal{Z}_R(\tau)^a{}_{b-a} \quad \text{and} \quad \mathcal{Z}_R(-1/\tau)^a{}_b = \mathcal{Z}_R(\tau)^b{}_{-a}\,. \tag{1.3}$$

Modular covariance is a stronger condition than modular invariance, but is often much easier to check. It is easy to convince oneself that combining two sets of sectors which are individually modular covariant will always lead to a modular covariant, and in turn a modular invariant, theory. This leads to a great simplification in these constructions, since the modular invariance of an interpolating orbifold will be guaranteed if the pure orbifold, the orbifold without the coordinate shift, is modular covariant.

In section 2 we will discuss the simplest example of a nongeometric monodromy: a circle fibered over another circle in which the fiber undergoes T-duality. This example is not modular invariant, but it serves as a useful illustration of where the consistency can fail. In section 3 we move on to the case of a 2-torus fiber, making contact with the work in [2]. We will check that the orbifold construction does indeed correspond to the spacetime constructions discussed in [2], and then, in order to check quantum consistency, we will explicitly construct the partition function and check modular invariance. As will be shown, these particular asymmetric orbifolds are dual to symmetric orbifolds, connecting these seemingly exotic backgrounds to more traditional compactifications. In section 4, we'll consider a more general torus fibration in the context of the heterotic string. It will be shown that, satisfying certain constraints on dimensionality, one can construct modular invariant interpolating asymmetric orbifolds which are truly, in the sense that they do not have a geometric dual, nongeometric. As a final example, in section 5 we construct interpolating orbifold realizations of monodrofolds involving modular shifts. In doing so we give a simple example of how to construct interpolating orbifolds when the corresponding pure orbifold is itself a product orbifold, and discuss the subtleties which subsequently arise.

2. The circle twist and its discontents

Let's consider the most trivial nongeometric twist. Namely, a circle (at the $SU(2)$ radius) fibered over another circle, where the fiber circle undergoes a T-duality transformation. From the worldsheet perspective this can be understood as an orbifolding by the action

$$\begin{aligned} X_R &\to -X_R \\ X_L &\to X_L \\ Y &\to Y + 2\pi R\,. \end{aligned} \tag{2.1}$$

Here X/Y is the embedding coordinate corresponding to the fiber/base circle.

As discussed in the introduction, the quantum consistency of (2.1) is dependent upon the modular covariance of the orbifold without the shift,

$$X_R \to -X_R\,, \quad X_L \to X_L\,. \tag{2.2}$$

[3]This is shown in appendix A.

Its not hard to see that the theory corresponding to (2.2) is not modular invariant, much less modular covariant. Consider the partition function for a symmetric reflection orbifold,

$$Z_{S^1/R} = \frac{1}{2} Z_{S^1} + \left| \frac{\eta(\tau)}{\vartheta_{10}(\tau)} \right| + \left| \frac{\eta(\tau)}{\vartheta_{01}(\tau)} \right| + \left| \frac{\eta(\tau)}{\vartheta_{00}(\tau)} \right| . \tag{2.3}$$

Here $\eta(\tau)$ are the Dedekind eta functions and $\vartheta_{\alpha\beta}(\tau) \equiv \theta_{\alpha\beta}(0,\tau)$, where $\theta_{\alpha\beta}(\nu,\tau)$ are the theta functions with characteristics. It is straightforward to show that, up to phases, the $\eta/\vartheta_{\alpha\beta}$ transform into one another under modular transformations. Since (2.3) contains the absolute value of these terms the phases cancel and the entire expression is modular invariant. Now consider the asymmetric reflection (2.2), which we will also refer to as a chiral reflection. From what we've learned in the symmetric case one might naively write down the expression,

$$Z^{\mathrm{NAIVE}}_{S^1/T} = \frac{1}{2} Z_{S^1} + \left(\frac{\eta(\tau)}{\vartheta_{10}(\tau)} \right)^{\frac{1}{2}} Z^r_{S^1} + \left(\frac{\eta(\tau)}{\vartheta_{01}(\tau)} \right)^{\frac{1}{2}} Z^r_{S^1} + \left(\frac{\eta(\tau)}{\vartheta_{00}(\tau)} \right)^{\frac{1}{2}} Z^r_{S^1} \tag{2.4}$$

where $Z^r_{S^1}$ is the contribution to the S^1 partition function from the right movers. Though we have not explicitly shown what $Z^r_{S^1}$ is, it should be clear that the trivial cancellation of phases which took place in (2.3) will not occur in (2.4). With a little work one can show that (2.4) is not modular invariant.

With the simplest possible asymmetric orbifold failing one may begin to worry. However, there are many examples of modular invariant asymmetric orbifolds: Orbifolds involving chiral reflections (and chiral shifts) are precisely the subject of [26]. There it was shown that not only was the above expression not modular invariant, it was the wrong expression altogether. As discussed in [26], our intuition on how to construct this orbifold breaks down due to a phase ambiguity that arises when splitting up the rotation $R(\theta)$ into a left/right-handed piece; $R(\theta) = R_L(\theta)R_R(\theta)$. In particular they show that the chiral reflection is 4th (rather than 2nd) order, and that one must be in $4m \in 4\mathbb{Z}$ dimensions if one hopes to have a modular invariant theory. There are also more elaborate examples. For example in [27–29] it was shown that starting with a T^4 at the $SO(8)$ point and combining chiral reflections of four coordinates with half-shifts along each T^4 cycle results in a modular invariant theory.

Interpolating orbifolds, Wilson lines and quantum consistency. Before moving on to modular invariant orbifolds we would like to address one final issue concerning (2.1). It might seem surprising that (2.1) is not consistent, since it looks very much like turning on a Wilson line. To review: At the self-dual radius our symmetry group is enhanced from $U(1)_L \times U(1)_R$ to $SU(2)_L \times SU(2)_R$. The generators for the $SU(2)_L$ are,

$$j^+_X = \exp\left(+i2X_L/\sqrt{\alpha'}\right) \tag{2.5}$$

$$j^-_X = \exp\left(-i2X_L/\sqrt{\alpha'}\right) \tag{2.6}$$

$$j^3_X = \partial X \, , \tag{2.7}$$

and similarly for $SU(2)_R$. One is then able to form vertex operators which lead to marginal deformations of our worldsheet theory. One may think of these deformations in terms of

Wilson lines since,

$$\int e^S \to \int e^S \exp\left(\int d^2\sigma(\partial Y \alpha \cdot \bar{j}_X + \bar{\partial} Y \beta \cdot j_X)\right) \tag{2.8}$$

$$= \int e^S \exp\left(\int d^{10} X^\mu (\bar{A}_\mu + A_\mu)\right) . \tag{2.9}$$

Where α^a, β^a are constant vectors whose indices run over $a = +, -, 3$,

$$A_Y \equiv \int dz\ \beta \cdot j_X \quad \text{and} \quad \bar{A}_Y \equiv \int d\bar{z}\ \alpha \cdot \bar{j}_X\,, \tag{2.10}$$

and all other components of A_μ vanish. To see how these Wilson lines may affect boundary conditions consider deforming the theory by the operator,

$$\frac{1}{4}\partial Y \bar{\partial} Y + \frac{1}{2}(\partial Y \bar{\partial} X + \partial X \bar{\partial} Y)\,. \tag{2.11}$$

This is of course nothing more than a metric deformation. To be precise, if we started with X, Y being the cycles of a T^2 with $\tau = i$, the deformed theory would have $\tau = 1/2 + i$. This skewed torus, is then just a S^1 fibered over a S^1 where the fiber shifts half way around as it circles the base.

Let's now return to our original question: is there a Wilson line which we can turn on which corresponds to imposing a chiral rotation as a boundary condition? Using our $SU(2)$ symmetry we can ask an equivalent, but more transparent question. Is it possible to turn on an operator which corresponds to imposing a chiral shift boundary condition? Using what we've learned from (2.11) we can see that such an operation should involve $\bar{j}^3$. For instance one might suggest $\partial Y \bar{\partial} X + \bar{\partial} Y \bar{\partial} X$, but one must not forget that we are looking for marginal deformations, i.e. weight $(1,1)$. The $\bar{\partial} Y \bar{\partial} X$ piece breaks conformal invariance. A more plausible suggestion would be $(1/4)\partial Y \bar{\partial} Y + \partial Y \bar{\partial} X$. However, it is not difficult to show that this simply corresponds to a symmetric background with $\tau = \rho = 1/2 + i$. In fact any linear combination of marginal operators involving $\partial Y, \bar{\partial} Y, j^3, \bar{j}^3$ will amount to nothing more than skewing the torus with a constant B-field. Moreover, since we are only interested in shifts, the operators $j^\pm, \bar{j}^\pm$ do not play a role. Our claim is then, although there are elements of $SU(2)_L \times SU(2)_R$ which correspond to chiral shifts or rotations, there are no $(1,1)$ operators which correspond to imposing these boundary conditions. Moreover, we believe, though we have certainly have not proved it here, that all marginal deformations of the world sheet theory are geometric, and therefore there is no way to deform from any geometric theory into any truly nongeometric theory, and vice versa.

3. Symmetric and pseudo asymmetric orbifolds

We would now like to turn our attention to backgrounds which are modular invariant. There are two classes which we know will work: backgrounds which are dual to geometric backgrounds, and backgrounds which do not have a geometric dual but come from a consistent asymmetric orbifold. Distinguishing truly nongeometric compactifications from those which are dual to geometric compactifications can sometimes be tricky. However, the spacetime picture can help.

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Consider a T^2, with a complex structure τ and the kahler modulus ρ, fibered over a circle. The symmetry group of the torus is $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$, where the $SL(2,\mathbb{Z})_\tau$ is the geometric symmetry group of the torus and $SL(2,\mathbb{Z})_\rho$ comes from T-duality and B-field shifts. If one looks at a monodromy $\rho \to -1/\rho$ the background naively looks nongeometric. However, T-dualizing on one of the torus cycles exchanges ρ and τ, and therefore, in its T-dual picture, the background is geometric. We will refer to such backgrounds as being pseudo asymmetric (nongeometric). Now consider the monodromy $\tau \to -1/\tau,\ \rho \to -1/\rho$. Because of the symmetry of the ρ and τ monodromies, anything we do to make the nongeometric monodromy geometric will also make the geometric monodromy nongeometric. Such a background does not have a dual geometric interpretation, and will be referred to as truly asymmetric (nongeometric). In the rest of this section we will discuss a symmetric orbifold and its pseudo asymmetric dual. In the next section we'll return to the discussion of truly asymmetric orbifolds.[4]

The focus of [2] is on the spacetime construction for T^2 fibrations over a T^2 base. In particular it looks at backgrounds where the T^2 fiber undergoes a nongeometric twist. The supersymmetry transformations require

$$\bar\partial\rho = \bar\partial\tau = \partial\bar\partial(\ln\tilde\rho_2 - \ln\rho_2 - \ln\tau_2) = 0\,. \tag{3.1}$$

Here τ ($\tilde\tau$) and ρ ($\tilde\rho$) denote the complex structure and Kähler moduli of the fiber (base), respectively. Generically such solutions preserve (1,0) supersymmetry in six dimensions. For the type IIA theory, solutions with constant ρ preserve (1,1) supersymmetry and solutions with constant τ preserve (2,0) supersymmetry. It follows from (3.1) that there is a nontrivial backreaction on the base which forces the corresponding fiber moduli to be periodic. If one then imposes nontrivial (not periodic) boundary conditions the moduli must take values which are fixed under the action of the monodromy. For completeness, the details of this argument as originally presented in [2] are repeated in appendix C.

As an example consider type II A with the monodromy:

$$\begin{array}{ll} \rho \to \rho\,,\ \tau \to -1/\tau & (\tilde\theta^1 - cycle) \\ \rho \to -1/\rho\,,\ \tau \to \tau & (\tilde\theta^2 - cycle)\ . \end{array} \tag{3.2}$$

It follows that τ and ρ are fixed to the values $\rho = \tau = i$. The fiber moduli coming from the NSNS sector are projected out by the monodromy (3.2).[5]

The lifting of moduli with boundary conditions is the central point of these constructions. If one were only to consider the values of the 10-dimensional metric and B-field they would see nothing more than a flat metric with vanishing B-field. However, (3.2) contains additional information, namely, the boundary conditions of the fluctuations about this background. These boundary conditions manifest themselves in the field content of the dimensionally reduced theory. In particular, for (3.2) there are no massless six-dimensional

[4] Here we have argued that there is no dual coming from the perturbative string duality group, one should really consider the entire U-duality group. This being said we don't suspect that any geometric dual will exist.

[5] Additionally, half of the RR moduli are lifted [2].

fields coming from NSNS moduli of the T^2 fiber. It is worth noting that even in traditional compactifications one is specifying (periodic) boundary conditions. However there is no reason from a string's perspective to place preference on periodic boundary conditions over other ones. It is certainly an interesting question to ask if there is some dynamical, thermodynamic or other way in which a string could "choose" one boundary condition over another.

The implications of nontrivial boundary conditions on the low energy spacetime physics was discussed in detail in [2]. The rest of this paper discusses these same backgrounds from a worldsheet perspective. In constructing the worldsheet theory we will be able to study the full spectrum, both massless and massive, and check quantum consistency.

As discussed above, we will construct these backgrounds in terms of interpolating orbifolds. The nongeometric nature of these compactifications will be reflected in the fact that the orbifolds themselves are asymmetric. A great deal of work has gone into the construction of consistent asymmetric orbifolds [30, 31, 26]. It is a very subtle procedure which one would like to avoid if at all possible. Below we will discuss a pseudo asymmetric orbifold. Since modular invariance is guaranteed from the dual symmetric orbifold, the discussion is greatly simplified. In the next section we will move on to a truly asymmetric example.

Let us first consider a spacetime in which the T^2 fiber undergoes a monodromy $\tau \to -1/\tau$ as it traverses a circle of radius $4R$, S^1_{4R}. From the worldsheet perspective this corresponds to orbifolding the theory on a $T^2 \times S^1_{4R}$ by the action,

$$\begin{aligned} X^1 &\to -X^2 \\ X^2 &\to X^1 \\ Y &\to Y + 2\pi R\,. \end{aligned} \tag{3.3}$$

Here X^1 and X^2 are the periodic bosons of the T^2 and Y is the boson coming from the S^1_{4R} (a circle of radius $4R$). The dual pseudo nongeometric monodromy is $\rho \to -1/\rho$. This corresponds to two T-dualities and a 90 degree rotation. The orbifold action is then

$$\begin{gathered} X^1_L \to -X^2_L \quad , \quad X^2_R \to -X^1_R \\ X^2_L \to X^1_L \quad , \quad X^1_R \to X^2_R \\ Y \to Y + 2\pi R\,. \end{gathered} \tag{3.4}$$

For the type II string, modular invariance is guaranteed since (3.3,3.4) is a symmetric/pseudo-asymmetric orbifold. For completeness we will write down the partition functions below. However, we should first check that orbifolding by (3.3) does indeed project out the same moduli as the corresponding monodrofold.

Moduli fixing. From the worldsheet perspective giving a mass to fluctuations in the metric and/or B-field corresponds to projecting out particular massless closed string states. We'll discuss the fate of these states below, but first we must build our Hilbert space. As usual, we quantize the system around a flat background (namely $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu$, $B = 0$)

and then build closed string states by acting repeatedly with the operators $\alpha^{\mu}_{-n}\tilde{\alpha}^{\nu}_{-m}$.[6] The states are labeled by the occupation numbers $(N_i, \widetilde{N}_i)$ and zero modes (n_i, w_i) associated with the fields X^i and Y, namely,

$$|N_1, \widetilde{N}_1, n_1, w_1; N_2, \widetilde{N}_2, n_2, w_2; N_Y, \widetilde{N}_Y, n_Y, w_Y; \cdots\rangle\,. \tag{3.5}$$

The $\cdots$ represents the eigenvalues coming from the other fields in the theory; they will henceforth be left out. Under (3.3) we find that (3.5) becomes

$$(e^{\pi i/2})^{n_Y}(-)^{N_1+\widetilde{N}_1}|\{2\};\{1\};\{Y\}\rangle\,. \tag{3.6}$$

Here we have used the short hand notation $\{i\} = N_i, \widetilde{N}_i, n_i, w_i$.

At the massless level $n_Y = 0 \Rightarrow i^{n_Y} = 1$. The quarter-shift around the circle, therefore, does not play a role in determining the fate of the moduli; for this reason we will drop the $\{Y\}$. Similarly zero modes n_i, w_i will be dropped. Note that the massive states do transform non-trivially under the quarter-shift. As will be seen when we construct the partition functions, the theory with the shift and without the shift are very different. Moreover, it is worth noting that here we are only discussing the untwisted sector. One might be concerned that the twisted sector includes new massless states. Indeed, this is often the case for orbifolds which are not integrated with a shift. However, as we will see explicitly below, integrating the shift with the pure orbifold gives masses to all twisted sector states.[7]

Fluctuations in the metric/B-field correspond to states created by acting with the symmetric/antisymmetric combinations of $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}$. To see which states are projected out we first form the eigenstates

$$|\pm;\{1\},\{2\}\rangle \equiv |\{1\},\{2\}\rangle \pm |\{2\},\{1\}\rangle\,. \tag{3.7}$$

It is easy to convince oneself that at the massless level $N_1 + \widetilde{N}_1 = N_2 + \widetilde{N}_2$ mod2 and (3.7) are then eigenstates with eigenvalues $\pm(-)^{N_1+\widetilde{N}_1}$. Under (3.3),

$$|\pm;1,1;0,0\rangle \longrightarrow \pm|\pm;1,1;0,0\rangle \tag{3.8}$$

$$|\pm;1,0;0,1\rangle \longrightarrow \mp|\pm;1,0;0,1\rangle\,. \tag{3.9}$$

The orbifold projection requires us to keep only the states with eigenvalue $+1$. From (3.8) we see that the fluctuations, h_{ij}, in the torus metric must satisfy $h_{11} = h_{22}$, and from (3.9) we see that $h_{12} = 0$. Fluctuations in the B-field are not projected out. Since we have quantized around $ds_{T^2} = \delta_{ij}dX^i dX^j$ this is exactly the condition $\tau = i$. In the dual (pseudo asymmetric) theory, where $\rho \to -1/\rho$, the action of the corresponding orbifold is (3.3) along with $X^i_R \to -X^i_R$. For states with $n_i = w_j = 0$,

$$|\{1\};\{2\};\{Y\}\rangle \underbrace{\longrightarrow}_{X^i_R\to -X^i_R} (-)^{\widetilde{N}_1+\widetilde{N}_2}|\{1\};\{2\};\{Y\}\rangle\,. \tag{3.10}$$

[6]Here we are discussing the bosonic string. In the Type II string there are slight modifications to the details of the argument, the bosonic oscillators are now world sheet fermions $\psi^{NS}_{-n\ \mu}, \tilde{\psi}^{NS}_{-n\ \nu}$ for instance, but the end result is the same.

[7]For this to be true one must be careful in the way the shift orbifold is constructed. The subtleties in how one constructs the shift orbifold will be discussed shortly.

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Noting that, at the massless level $(-)^{\widetilde{N}_1+\widetilde{N}_2} = -1$, we can see integrating $X^i_R \to -X^i_R$ with (3.3) introduces an overall change in sign in (3.8) and (3.9). As expected, projecting onto states with eigenvalue $+1$ gives condition $\rho = i$.

The partition function

We will now construct the partition function for these orbifolds. First consider orbifolding the torus by a 90 degree rotation (without the interpolation). The sector with a twists and b insertions[8] is given by ,

$$Z^a{}_b(\tau) = |\mathcal{Q}^a_b|^2 \ ; \tag{3.11}$$

$$\mathcal{Q}^a_b \equiv \frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}-\frac{a}{4}\\ \frac{1}{2}+\frac{b}{4}\end{bmatrix}(\tau)} \times \exp\left[2\pi i \left(\frac{b}{4}+\frac{1}{2}\right)\left(\frac{a}{4}-\frac{1}{2}\right)\right] \times (1-e^{\pi i b/2})^{\delta_{a,0}} \,. \tag{3.12}$$

Because the two theories are T-dual, the partition function for the $\rho \to -1/\rho$ monodrofold should be the same as the $\tau \to -1/\tau$ monodrofold. To check this explicitly one needs to construct the pure orbifold of the dual theory. This is a worthwhile exercise since, from the worldsheet perspective, there is a priori no reason to expect the two orbifolds to be dual. This gives further, albeit indirect, evidence that these interpolating orbifolds are in fact the correct worldsheet description of monodrofolds. This is easy to check. $\rho \to -1/\rho$ corresponds to T-dualizing along both legs of the torus along with a 90 degree rotation. In terms of our embedding coordinates this is simply a 90 degree rotation on the left hand side of the string and a -90 degree rotation on the right hand side of the string. Being careful with the phases one finds,

$$\widetilde{Z}^a{}_b(\tau) = \mathcal{Q}^a_b \bar{\mathcal{Q}}^{||-a||}_{||-b||} \tag{3.13}$$

$$= |\mathcal{Q}^a_b|^2 \tag{3.14}$$

The "$||m||$" in the superscripts and subscripts indicates that one should use the lowest positive value of "m"; i.e. m= 0,1,2,3 (for a fourth order orbifold). Going from (3.13) to (3.14) we have used known transformation properties of the ϑ functions. As expected, (3.14) agrees with (3.11). Its important that one not over extend the validity of the expression (3.12). $\mathcal{Q}^a_b$ ($\bar{\mathcal{Q}}^a_b$) is the contribution to the partition function coming from the left (right) side of the string. However the derivation only holds for the symmetric and pseudo asymmetric orbifolds, and one can not arbitrarily pair a $\mathcal{Q}^a_b$ with a $\bar{\mathcal{Q}}^c_d$ to form asymmetric orbifold sectors. For instance, in the orbifolds discussed in this subsection there are no terms $\mathcal{Q}^2_b\bar{\mathcal{Q}}^0_0$. Such a sector would seemingly correspond to twisting the left hand side of the string by 180 degrees while leaving the right hand side of the string unaffected. As discussed in section 2, this is the incorrect partition trace for such a orbifold, and moreover, such an orbifold is not modular invariant.

[8]This does not hold for $a = b = 0$ since we have not included zero modes. Its also worth noting that $\mathcal{Q}^a_b \propto 1/Z^{1-a/2}_{1+b/2}$ where $Z^{1-a/2}_{1+b/2}$ is defined in (B.18).

If one were not to integrate the $\tau \to -1/\tau$ (or $\rho \to -1/\rho$) orbifold with a shift around the base, the total partition function would be

$$Z(\tau) = iV_{10} \int_F \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} Z^{[10]}(\tau) \; ; \tag{3.15}$$

$$Z^{[10]}(\tau) \equiv Z^{[6]}(\tau)\; \mathcal{Z}_R(\tau)^2 \; \frac{1}{4}\sum_{a,b} Z^a{}_b(\tau)\,. \tag{3.16}$$

$Z^{[6]}(\tau)$ is the partition function trace for the fields which do not play a role in (3.2) and $\mathcal{Z}_R(\tau)$ is the partition function for a circle of radius R;

$$\mathcal{Z}_R(\tau) = |\eta(\tau)|^{-2} \sum_{n,w=\infty}^{\infty} \exp\left[-\pi\tau_2\left(\frac{\alpha' n^2}{R^2} + \frac{w^2R^2}{\alpha'}\right) + 2\pi i\tau_1 nw\right]. \tag{3.17}$$

In the partition function for the interpolating orbifold, the $\{a,b\}$ contribution to the shift orbifold must be paired up with $\{a,b\}$ contribution to the pure orbifold. Namely,

$$Z^{[10]}(\tau) = Z^{[6]}(\tau)\mathcal{Z}_R(\tau)\; \frac{1}{4}\sum_{a,b} Z^a{}_b(\tau)\; \mathcal{Z}_R(\tau)^a{}_b\,. \tag{3.18}$$

Where,

$$\mathcal{Z}_R(\tau) = \frac{1}{4}\sum_{a,b} \mathcal{Z}_R(\tau)^a{}_b\,. \tag{3.19}$$

As suggested above, to construct the $\mathcal{Z}_R(\tau)^a{}_b$ traces we will start with a circle of radius $4R$ and orbifold by a quarter-shift around the circle. Orbifolding by the shift alone simply quarters the size of circle, $\mathcal{Z}_{4R}(\tau) \to \mathcal{Z}_R(\tau)$. It would certainly be possible to choose an Nth-order shift around a circle of radius NR. Indeed, from what we have already learned, the massless sector coming from the untwisted sector is unaffected by the shift. Moreover, as long as both the shift orbifold and the pure orbifold are modular covariant the theory will be modular invariant. However, in order to make connection with the spacetime constructions in [2] we argue that an order N pure orbifold action must be integrated with an order N shift. To see this we should return to the interpolating nature of the backgrounds discussed in [2]. There, there are masses given to certain fluctuations by introducing nonstandard boundary conditions. However, as one takes the the base to infinite volume the fields once again become massless. Lets now try and follow this same story from the worldsheet perspective:

We will start by constructing the partition traces (which we needed to do anyway). Consider a Nth order orbifold, namely a $2\pi R$ shift on a circle of radius NR. It is easy to reason out what the $\mathcal{Z}_R{}^a{}_b$ should be: $\mathcal{Z}_R{}^0{}_0 = \mathcal{Z}_{NR}[n|w]$ by definition. The funny notation "$[n|w]$" is used to indicate that the trace sums over integers n and w. For example $\mathcal{Z}_{NR}[n|w]$, which is equal to (3.17) after replacing R with NR, contains a sum over all integer values of n. However, $\mathcal{Z}_{NR}[2n|w]$ only sums over even integers,[9] $2n \in 2\mathbb{Z}$. In order to find $\mathcal{Z}_R{}^0{}_b$ we need to insert the operator $\exp(2\pi ibn/N)$ into the sum (3.17). It follows

[9]This notation is explained further in appendix A.

that,

$$\mathcal{Z}_R{}^0{}_b = \sum_{q=0}^{N-1} e^{2\pi i b q/N} \mathcal{Z}_{NR}[Nn+q|w] \tag{3.20}$$

In the a-th twisted sector, $\mathcal{Z}_R{}^a{}_0$, we now have strings which wind a/N of the way around the original circle. It follows that one must replace w with $w + a/N$,

$$\mathcal{Z}_R{}^a{}_0 = \mathcal{Z}_{NR}[n|w+a/N]\,. \tag{3.21}$$

Combining these operations is straightforward. We find that the sector with a twists and b insertions is given by the trace

$$\mathcal{Z}_R{}^a{}_b = \sum_{q=0}^{N-1} e^{2\pi i b q/N} \mathcal{Z}_{NR}[Nn+q|w+a/N] \ \ . \tag{3.22}$$

As a quick check note that,

$$\begin{aligned}
\frac{1}{N}\sum_{a,b} \mathcal{Z}_R{}^a{}_b &= \frac{1}{N}\sum_{q,a} \underbrace{\left[\sum_b e^{2\pi i b q/N}\right]}_{=N\delta_{q,0}} \mathcal{Z}_{NR}[Nn+q|w+a/N] \\
&= \sum_a \mathcal{Z}_{NR}[Nn|w+a/N] \\
&= \mathcal{Z}_{NR}[Nn|w/N] \\
&= \mathcal{Z}_R \ \ .
\end{aligned} \tag{3.23}$$

Having constructed the partition traces it is now easy to see why, when integrating a shift with an Nth order pure orbifold, we require an order N twist. Suppose for a-twists $a/N = m \in \mathbb{Z}$. The twisted winding states in the a-th twisted sectors now include a massless mode, since $w \to w + m \in \mathbb{Z}$. Taking $R \to \infty$ the $m = 0$ mode does not drop out of the theory. It follows that there exist twisted states, *i.e.* states which do not exist in the unorbifolded theory, which do not drop out of the theory as we go to the noncompact limit. Such an orbifold is not, therefore, "interpolating" in the sense presented above.[10] Having said this, we have constructed the shift orbifold for general N, and in turn given a world sheet description to our pure orbifold integrated with an Nth order shift. It would be interesting to try to identify the the corresponding spacetime physics and contrast it with the monodrofold discussed above.

Using (3.11,3.17,3.18,3.22) one obtains the full partition function for a $2\pi/4$ rotation integrated with a quarter-shift around the circle. Modular covariance of (3.11,3.14) follows trivially from transformation properties of the η/ϑ-functions. The modular covariance of the $\mathcal{Z}_R{}^a{}_b$ is demonstrated in appendix A. As discussed in section 2, this guarantees the modular invariance of the total partition function (3.18).

[10] We have been a little to slick in this argument, and in the next section will have too slightly modify our conclusions.

4. Truly asymmetric orbifolds

Lets now turn to the heterotic string. In particular we will focus on the $SO(32)$ case, though everything can be easily generalized to the $E_8 \times E_8$ case. The construction in this section involves an asymmetric orbifold which is truly nongeometric. We'll see that although from the spacetime perspective this is possible in an arbitrary number of dimensions, modular invariance forces us to have either be in 4 or 8 dimensions. To do this we will have to go to higher dimensional fibrations, which is convenient since one is ultimately interested in getting down to 4-dimensional theories. To do this we will make a trivial modification of the the backgrounds discussed in [2]. Rather than looking at T^2 fibrations over a $T^2 \times T^{16}$ base we will consider $T^2 \times T^2 \times T^{16}$ fibrations over a T^2 base. Here T^{16} is the internal torus of the heterotic string. (3.1) is then modified in the obvious way,

$$\bar{\partial}\rho^i = \bar{\partial}\tau^i = \partial\bar{\partial}(\ln\tilde{\rho}_2 - \ln\rho_2^1 - \ln\tau_2^1 - \ln\rho_2^2 - \ln\tau_2^2) = 0\,. \tag{4.1}$$

Here ρ^i, τ^i correspond to the moduli coming from the $i = 1, 2$ torus. In addition to (4.1) there is a self-duality constraint on field strength arising from the internal gauge fields. As usual we require the gauge fields commute so that the potential $\sim Tr([A_\mu, A_\nu]^2)$ vanishes, thus the most general Wilson line background uses only the sixteen $U(1)$ gauge fields in the Cartan of the heterotic gauge group. Each of these $U(1)$'s can be associated with a cycle of the internal torus. Requiring that they are constant along the T^2 fiber directions, the self-duality constraint reduces to the fact that the gauge fields must also be constant along the base T^2 [2], i.e.

$$\partial_\mu A_\nu^{\mathcal{J}} = 0\,. \tag{4.2}$$

In particular we will look at the monodromy

$$\tau^i \to -1/\tau^i\,, \quad \rho^i \to -1/\rho^i\,, \quad T^{16} \to \mathcal{R}_n T^{16}. \tag{4.3}$$

as we encircle one of the cycles in the base. Here $\mathcal{R}_n$ is a reflection on n of the internal bosons, or in the fermionic picture, n of the internal complex fermions. As discussed in the beginning of section 3 this corresponds to a truly nongeometric background.

Moduli fixing. As before, the spacetime boundary conditions fix $\rho^i = \tau^i = i$. The boundary conditions also require $T^{16} \to \mathcal{R}_n T^{16}$. Since the gauge fields must be constant (4.2) this forces them to vanish on the directions in which $\mathcal{R}_n$ has a nontrivial action.

From the worldsheet perspective the orbifold projection is quite similar to that in the previous section. The usual geometric and B-field moduli in the heterotic string arise from symmetric/antisymmetric combinations of $\alpha_{-1\ \mu}\tilde{\psi}^{NS}_{-1/2\ \nu}$ acting on the NS vacua. In addition we have the moduli from heterotic gauge fields which lie in the Cartan of $SO(32)$ or $E_8 \times E_8$ with constant expectation values along the T^2 fibers. In the bosonic formulation these states arise from $\alpha^i_{-1}\tilde{\alpha}^\mu_{-1}$ where i labels the Cartan $U(1)$ factors. Orbifolding by a reflection in the ith cycle of the internal torus introduces $(-1)^N_i$ under which $\alpha^i_{-1}\tilde{\alpha}^\mu_{-1}$ is odd. As expected from the spacetime physics, the states corresponding to the massless fluctuations of $A^{\mathcal{J}}_\nu$ are projected out of the theory.

The partition function. In order to construct the world sheet theory we follow the same steps as in the previous section. Namely, we first construct the partition traces for the pure orbifold and show they are modular covariant. We then integrate the pure orbifold with a shift around the circle.

The internal and fermionic contribution to the partition function for the heterotic string is $Z_{16}Z_{\psi}^{+*}$ [32]. Here Z_{16} is the contribution coming from the internal lattice and Z_{ψ}^{+*} is the contribution coming from the fermions. In terms of Z_{β}^{α}, as defined in (B.18),

$$Z_{16}(\tau) = \frac{1}{2}[Z_0^0(\tau)^{16} + Z_1^0(\tau)^{16} + Z_0^1(\tau)^{16} + Z_1^1(\tau)^{16}] \tag{4.4}$$

$$Z_{\psi}^{+}(\tau) = \frac{1}{2}[Z_0^0(\tau)^{4} - Z_1^0(\tau)^{4} - Z_0^1(\tau)^{4} - Z_1^1(\tau)^{4}]\,. \tag{4.5}$$

Its worth noting that although the product $Z_{16}Z_{\psi}^{+*}$ is modular invariant, the individual partition sums (4.4,4.5) are not.[11] To be precise, under modular shifts $\tau \to \tau + 1$

$$Z_{16}(\tau+1) \to e^{2\pi i/3} Z_{16}(\tau)\,, \quad Z_{\psi}^{+}(\tau+1) \to e^{2\pi i/3} Z_{\psi}^{+}(\tau)\,. \tag{4.6}$$

This gives us a hint on what to look for as we proceed. We will focus on asymmetric reflection orbifolds of the embedding coordinates and the internal lattice. Since these orbifolds will leave the fermionic contribution to the partition sum unaffected, the partition sum coming from orbifolded piece must pick up a phase under modular shifts. Keeping this in mind we will relax the condition of modular covariance, and look for theories which are modular covariant up to phases.

First consider reflection orbifolds of the internal lattice. The details are worked through in appendix B. Starting with the α,β-sector, $Z_{\beta}^{\alpha\ 16}$, and preforming a twists and b insertions on n of the dimensions one finds

$$Z_{\beta}^{\alpha\ 16} \longrightarrow (-)^{\alpha(\alpha-1)\beta/2} Z_{\beta}^{\alpha\ 16-n} Z_{\beta+b}^{\alpha-a\ n}\,. \tag{4.7}$$

The total partition function for the internal part is then

$$Z_{16/\mathcal{R}} = \sum_{a,b=0}^{3} \lambda_b^a\,, \tag{4.8}$$

where

$$\lambda_b^a \equiv \sum_{\alpha,\beta=0}^{1} (\mathrm{sgn}_{ab}[\alpha,\beta])^n Z_{\beta}^{\alpha\ 16-n} Z_{||\beta+b||}^{||\alpha-a||\ n}\,. \tag{4.9}$$

Here $||m|| \equiv (1-(-)^m)/2$, *i.e.* 0 if m is even and 1 if m is odd.[12] Table 1 gives the values for $\mathrm{sgn}_{ab}[\alpha,\beta]$. When orbifolding an even number of dimensions, $n \in 2\mathbb{Z}$ there are only 4

[11] Under inversions, $\tau \to -1/\tau$, the partition sums do transform into themselves.

[12] Though we are using the same notation "$||\ldots||$" as in section 3 they have a slightly different meaning. Here we are restricting ourselves to $m = 0, 1$ even though we are discussing a 4th order orbifold.

a	0	0	2	2	0	0	2	2	1	1	3	3	1	1	3	3
b	0	2	0	2	1	3	1	3	0	2	0	2	1	3	1	3
α, β																
0,0	+	+	+	+	+	+	-	-	+	-	+	-	-	+	+	-
0,1	+	+	-	-	+	+	+	+	-	+	+	-	-	+	-	+
1,0	+	-	+	-	+	-	-	+	+	+	+	+	+	+	-	-
1,1	+	-	-	+	-	+	-	+	+	+	-	-	+	+	+	+

Table 1: $\mathrm{sgn}_{ab}[\alpha, \beta]$: The a,b-columns have been subdivided into groups which have the same value for $||a||$ and $||b||$.

distinct sectors,

$$\lambda_0^0 = 2Z_{16} \tag{4.10}$$
$$\lambda_1^0 = Z_0^{0\ 16-n} Z_1^{0\ n} + Z_1^{0\ 16-n} Z_0^{0\ n} + Z_0^{1\ 16-n} Z_1^{1\ n} + Z_1^{1\ 16-n} Z_0^{1\ n} \tag{4.11}$$
$$\lambda_0^1 = Z_0^{0\ 16-n} Z_0^{1\ n} + Z_1^{0\ 16-n} Z_1^{1\ n} + Z_0^{1\ 16-n} Z_0^{0\ n} + Z_1^{1\ 16-n} Z_1^{0\ n} \tag{4.12}$$
$$\lambda_1^1 = Z_0^{0\ 16-n} Z_1^{1\ n} + Z_1^{0\ 16-n} Z_0^{1\ n} + Z_0^{1\ 16-n} Z_1^{0\ n} + Z_1^{1\ 16-n} Z_0^{0\ n} \tag{4.13}$$

Its easy to show that for $n = 4m \in 4\mathbb{Z}$ under $\tau \to -1/\tau$

$$\lambda_0^0 \to \lambda_0^0\,, \quad \lambda_1^0 \to \lambda_0^1\,, \quad \lambda_0^1 \to \lambda_1^0\,, \quad \lambda_1^1 \to \lambda_1^1\,, \tag{4.14}$$

and under $\tau \to \tau + 1$

$$\lambda_0^0 \to e^{2\pi i/3}\lambda_0^0\,, \quad \lambda_1^0 \to e^{2\pi i/3}\lambda_1^0\,, \quad \lambda_0^1 \to e^{2\pi i/3}(-)^m\lambda_1^1\,, \quad \lambda_1^1 \to e^{2\pi i/3}(-)^m\lambda_0^1\,. \tag{4.15}$$

Note that each term picks up a phase $e^{2\pi i/3}$. This is good news! However, in the λ_b^1 sectors there is an additional factor of $(-)^m$. If one were not to orbifold any of the worldsheet bosons it would be necessary to pick $m \in 2\mathbb{Z}_+$ or $n = 8, 16$. It turns out that even after orbifolding the the embedding coordinates this will be the case. This is certainly not surprising since we know that even self dual Euclidean lattices only exist in $8\mathbb{Z}_+$-dimensions.

Now we would like to orbifold d of the embedding coordinates by a chiral reflection. Fortunately this has been worked out for us in [26].[13] Denote the trace with a twists and b insertions by $\mathcal{K}_b^a$ and define,

$$Z_{X_b^a} = \left(\mathcal{K}_b^a\right)^d + \left(\mathcal{K}_b^{a+2}\right)^d + \left(\mathcal{K}_{b+2}^a\right)^d + \left(\mathcal{K}_{b+2}^{a+2}\right)^d. \tag{4.16}$$

It is the $Z_{X_b^a}$ which multiply the sectors in (4.10). As in [26] we will express these in terms of ζ-functions. There is a brief review of these functions in appendix D and a more detailed

[13] Our $\mathcal{K}_b^a$ corresponds to their $\mathcal{Z}_{r_L^a}^{r_L^b}$. Note that the positions of the twist/insertion super/sub-scripts has been reversed.

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review in the appendices of [26].

$$
\begin{aligned}
Z_X{}^0_0 &= (|\zeta_{00}|^2 + |\zeta_{\frac{1}{2}0}|^2)^d + (\zeta_{00}\bar{\zeta}_{\frac{1}{2}0} + \zeta_{\frac{1}{2}0}\bar{\zeta}_{00})^d \\
&\quad + (|\zeta_{00}|^2 - |\zeta_{\frac{1}{2}0}|^2)^d + (\zeta_{00}\bar{\zeta}_{\frac{1}{2}0} - \zeta_{\frac{1}{2}0}\bar{\zeta}_{00})^d \qquad (4.17) \\
\frac{1}{2} Z_X{}^0_1 &= \zeta^d_{0\frac{1}{2}}\bar{\zeta}^d_{00} + e^{i\pi d/4}\zeta^d_{0\frac{1}{2}}\bar{\zeta}^d_{\frac{1}{2}0} \qquad (4.18) \\
\frac{1}{2} Z_X{}^1_0 &= \zeta^d_{\frac{1}{4}0}\left[(\bar{\zeta}_{00} + \bar{\zeta}_{\frac{1}{2}0})^d + e^{i\pi d/4}(\bar{\zeta}_{00} - \bar{\zeta}_{\frac{1}{2}0})^d\right] \qquad (4.19) \\
\frac{1}{2} Z_X{}^1_1 &= e^{i\pi d/8}\zeta^d_{\frac{1}{4}\frac{1}{2}}\left[(\bar{\zeta}_{00} + e^{-i\pi/2}\bar{\zeta}_{\frac{1}{2}0})^d + e^{i\pi d/4}(\bar{\zeta}_{00} - e^{-i\pi/2}\bar{\zeta}_{\frac{1}{2}0})^d\right] \qquad (4.20)
\end{aligned}
$$

Using the ζ-function transformation properties it is easy to show if $d = 4$, under $\tau \to -1/\tau$

$$
Z_X{}^0_0 \to Z_X{}^0_0\,, \quad Z_X{}^0_1 \to Z_X{}^1_0\,, \quad Z_X{}^1_0 \to Z_X{}^0_1\,, \quad Z_X{}^1_1 \to Z_X{}^1_1\,, \qquad (4.21)
$$

and that under $\tau \to \tau + 1$

$$
Z_X{}^0_0 \to Z_X{}^0_0\,, \quad Z_X{}^0_1 \to Z_X{}^0_1\,, \quad Z_X{}^1_0 \to Z_X{}^1_1\,, \quad Z_X{}^1_1 \to Z_X{}^1_0\,. \qquad (4.22)
$$

Using (4.14,4.15,4.21,4.22), and defining $Z^{[10-d]}$ to be the partition trace coming from the unorbifolded bosons, we find the total partition function,

$$
Z = iV_{10}\int_F \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} Z^{[10-d]} Z_\psi^{+*}(\lambda^0_0 Z_X{}^0_0 + \lambda^0_1 Z_X{}^0_1 + \lambda^1_0 Z_X{}^1_0 + \lambda^1_1 Z_X{}^1_1)\,, \qquad (4.23)
$$

is modular covariant for $n = 0, 8, 16$ $d = 0, 4, 8$.

We can now integrate this orbifold with a shift to build the partition function corresponding to the monodrofold (4.3). For truly asymmetric orbifolds there is a new subtlety which arises in choosing the order of the shift orbifold. Recall that we have argued that an order N action should be integrated with an order N shift. Our reasoning came from the fact that we wanted to make sure that in the $R \to \infty$ limit the twisted sectors dropped out of the theory. The orbifold currently being discussed at first sight appears to be 4th order. However, the spacetime monodromy appears to be second order, and moreover our naive concept of a reflection is second order. Recall that, as discussed in section 2, the 4th order nature of the chiral reflection is due to the phase ambiguity which arises when splitting up the rotation $R(\theta)$ up into a left/right-handed piece; $R(\theta) = R_L(\theta)R_R(\theta)$ [26]. With this in mind let's take another look at the order of this orbifold. For each piece of the partition trace in (4.4) the action is 4th order. However, when orbifolding you must act on all 4 sectors in (4.4). Summing these we see that (in $d = 4\mathbb{Z}$) the operation is actually second order. The asymmetric orbifold on the spacetime bosons is trickier. As derived, the sum in (4.16) was not forced on us at the outset, but rather a grouping due to the second order nature of the internal lattice reflection. Moreover, the individual elements, $\mathcal{K}^a_b$, are not themselves modular covariant. If we then try and integrate them with a 4th order shift we will not have a modular invariant theory. However, if we integrate with a 2nd order shift the terms again group together properly and the theory is modular invariant. The question we must ask is then, does the orbifold integrated with the 2nd order shift correspond to the spacetime monodrofold defined by (4.3)? This question is equivalent to asking if one is allowed to keep sectors, in the $R \to \infty$ limit, in which there has been a chiral rotation of 360 degrees. Its our belief that since from a purely spacetime perspective this action is trivial we can still identify this orbifold with the monodrofold given in (4.3). Note that this is very different than the cases discussed in the previous section where the order 2 twist has definite spacetime implications. We will return to this issue in the discussion section.

The total partition function for the interpolating orbifold corresponding to the spacetime defined by the monodromy (4.3) is then

$$Z = iV_{10} \int_F \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} Z^{[10-d]} \mathcal{Z}_R Z_\psi^{+*} \sum_{a,b=0}^{1} \lambda_b^a Z_X{}^a_b \mathcal{Z}_R{}^a{}_b\,. \tag{4.24}$$

Here $\mathcal{Z}_R{}^a{}_b$ is as defined in (3.22) for a second order shift, d (the number of embedding coordinate which are orbifolded) is $0, 4$ or 8. and the number of reflections on the internal lattice is $0, 8$ or 16.

5. Monodrofolds involving modular shifts

So far in this paper we have concentrated on monodromies in which $\rho \to -1/\rho$ and/or $\tau \to -1/\tau$. As a final example we would like to consider monodromies which include modular shifts, $\tau \to \tau + 1$ ($\rho \to \rho + 1$). A shift alone does not admit constant moduli as solutions and is therefore not in the class of backgrounds we are discussing.[14] We can however find monodromies with constant moduli solutions if we combine modular inversions with modular shifts. There is some subtlety to combining these actions. For example, there are two seemingly different monodromies arising from combining a single inversion ($R:\ \tau \to -1/\tau$) and a single shift ($S:\ \tau \to \tau + 1$):

$$RS:\ \tau \to -\frac{1}{\tau + 1}\,, \quad SR:\ \tau \to -\frac{1}{\tau} + 1\,. \tag{5.1}$$

However, let's be careful before proceeding. The constant solutions for these monodromies are, $\tau_\pm^{RS} \equiv -1 \pm i\sqrt{3}$ and $\tau_\pm^{SR} \equiv 1 \pm i\sqrt{3}$ respectively. Clearly $\tau_\pm^{RS} = -\tau_\mp^{SR}$, and therefore the constant modulus describes the same torus (in different fundamental domains). From the spacetime picture we see that the massless fluctuations around our background are the same for the RS theory and the SR theory. From the worldsheet we will be able to make a much stronger statement, namely any differences between the RS theory and the SR theory do not contribute to the partition trace. This is reflective of the fact that when one says an orbifold is abelian they mean that the point group (the group coming just from the rotation generators) is abelian. This is trivially true in these orbifolds since S, as will be discussed below, is a pure shift.

In constructing the orbifold in these two theories the operation corresponding to R is of course the same rotation by 90-degrees which we discussed in section 3. S, which takes $\tau \to \tau + 1$, is simply $\exp[2\pi R(p_1 + p_2)]$, where p^i is the momentum along the coordinates

[14] In the discussion section we will say more about monodromies which force the moduli to vary.

of the fiber, X^i. To make this clear, think of the torus fiber as a circle fibered over another circle. Orbifolding by a simultaneous shift in the fiber circle and the base circle gives a theory such that, upon translation around the base the circle, the fiber must shift by $2\pi R$. Another way to think about this is if you start with as square torus and orbifold along the diagonal this gives a skewed torus with τ_2 equal to the length of the diagonal (divided by 2π).

In order to insure that the partition function is modular invariant, we will once again demand that the orbifold without the shift is modular covariant. Here we must be careful in defining what we mean by modular covariance. Defining

$$Q\left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array}\right]$$

to be the sector with a S-twists, b S-insertions, α R-twists, and β R-insertions, the correct modular covariance requirement is,

$$Q\left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array}\right] \longrightarrow Q\left[\begin{array}{c|c} a & \alpha \\ b-a & \beta-\alpha \end{array}\right] \tag{5.2}$$

under $\tau \to \tau + 1$, and

$$Q\left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array}\right] \longrightarrow Q\left[\begin{array}{c|c} b & \beta \\ -a & -\alpha \end{array}\right] \tag{5.3}$$

under $\tau \to -1/\tau$.

It is trivial to construct our $Q[...]$'s using the traces given in (3.11) and (3.22). By definition,

$$Q\left[\begin{array}{c|c} a & 0 \\ b & 0 \end{array}\right] = \mathcal{Z}_{R_1}{}^a{}_b \, \mathcal{Z}_{R_2}{}^a{}_b \tag{5.4}$$

$$Q\left[\begin{array}{c|c} 0 & \alpha \\ 0 & \beta \end{array}\right] = |\mathcal{Q}^{\alpha}_{\beta}|^2 . \tag{5.5}$$

To construct the other sectors consider first acting with elements of R. For the twisted sectors there are no zero modes, and for the untwisted sectors with insertions the zero modes drop out of the trace. It follows in neither case does the orbifold by the shift contribute to the partition trace:

$$Q\left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array}\right] = |\mathcal{Q}^{\alpha}_{\beta}|^2 \qquad \text{[Except for } \alpha = \beta = 0] \tag{5.6}$$

If one were first to act with S the story would be similar. The operators in the untwisted sectors would have different eigenvalues (since you'd be acting with p^i's before the rotation) but since the only contribution to the partition trace comes from $p^i = 0$ this does not effect our final expression.

So far we have focused on a single shift and a single inversion. However the results are easy to generalize. Since $R^{2n} \propto 1$ orbifolds with an even number of 90-degree rotations will not admit constant τ solutions. Moreover, $R^{2m+1} = (-)^m R$. Replacing a single S with S^N simply involves replacing the radii R_i in (5.4) with NR_i. Since the orbifold actions corresponding to S and R are abelian (in the sense discussed above) $R^{2m+1}S^N$ is the most general monodromy admitting constant τ solutions.

We can now integrate this orbifold with shift in the straightforward way. As before we want all twisted sectors to be massive and therefore the order of the shift must be the sum of the order of R^{2m+1} and S^N orbifolds, *i.e.* 5. The total partition function for the orbifold corresponding to space times in which a T^2 fiber undergoes $2m+1$ modular inversions and N modular shifts is then

$$Z = (-)^m i V_{10} \int_F \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} Z^{[6]}(\tau) \sum_{a,b,\alpha,\beta} Q(\tau) \left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array} \right]_N \mathcal{Z}_R(\tau)^{a+\alpha}{}_{b+\beta}\,. \tag{5.7}$$

Where

$$Q(\tau) \left[\begin{array}{c|c} a & \alpha \\ b & \beta \end{array} \right]_N = \begin{cases} \mathcal{Z}_{NR_1}(\tau)^a{}_b \, \mathcal{Z}_{NR_2}(\tau)^a{}_b & \text{if } \alpha = \beta = 0 \\ |\mathcal{Q}(\tau)^\alpha_\beta|^2 & \text{otherwise} \end{cases}\,. \tag{5.8}$$

To construct the pseudo asymmetric dual the S insertions will involve winding rather than momentum, and the R insertions are exactly as discussed in section 2.

6. Discussion

A crucial part of the development of string theory lies in identifying, and exploring the nature of, elements which are not present in the classical point particle theory. Nongeometric backgrounds in particular serve as important clues to how we might move away from the classical formulations of geometry on which string theory is so dependent. It has been shown from the spacetime perspective that by allowing various moduli to undergo monodromies in the full stringy duality group one may construct a broad class of nongeometric backgrounds [1 − 3, 9]. That such backgrounds can be built by incorporating stringy symmetries with a somewhat conventional spacetime approach may be surprising, and one might expect that this approach renders certain aspects difficult to discern. Doubled formalisms [20, 21], as well as the work being done with $SU(3)$ structures [13 − 15] and generalized complex geometry [18, 19] are perhaps the first steps towards shedding light into some of these areas.

Another obvious approach (and the one taken in this paper) is through the worldsheet. In this paper, we have used orbifold methods to construct nongeometric backgrounds, and argued that they correspond to the spacetimes discussed in [1, 2]. More precisely, we make explicit through several examples the connection between interpolating orbifolds and spacetime duality twists. By constructing one-loop partition functions and checking modular invariance, it has become clear that generic nongeometric backgrounds arising from duality twists will not have simple orbifold constructions. On the other hand, we have been able to find several positive results. We have constructed what we called pseudo nongeometric backgrounds. These backgrounds are dual to geometric backgrounds and as a result inherit the obvious consistency in the dual formulation. Additionally we have been able to formulate backgrounds which are truly nongeometric in the sense of not being dual (within the context of the perturbative stringy duality group) to any geometric solution.

In no sense has this work been exhaustive. We have limited ourselves to interpolating orbifolds with relatively little inclusion of shift symmetries and/or non-abelian point groups. By allowing for a greater variety in the orbifold groups one should be able to construct many more consistent examples. At present there seems little one can say on the pattern of consistent nongeometric backgrounds in full. A key open problem is to classify on general grounds, as opposed to the example by example approach taken here, which duality twists lead to modular invariant theories.

All of the above constructions involve spacetime monodromies with fixed points. Requiring constant moduli solutions was necessary to insure that our base would be a torus. There is even stronger motivation for this restriction since if we allow the moduli to vary in these simple models we would no longer have a solution to the string equations of motion. This being said, in more complicated compactifications with additional fluxes turned on it should be possible to find spacetimes which exhibit monodromies without fixed points. In fact these are exactly the types of backgrounds discussed in [12]. It is reasonable to ask if there are orbifold constructions of these models. Consider a simple example of a "twisted torus" [33], where

$$ds^2 = dx^2 + dy^2 + (dz + Mx\ dy)^2 \tag{6.1}$$

and the B-field vanishes. Globally we have the identifications $(x, y, z) \simeq (x, y+1, z) \simeq (x, y, z+1)$ and

$$(x, y, z) \simeq (x+1, y, z - My)\,. \tag{6.2}$$

The y-dependent shift in (6.2) is needed to make the torus metric is single valued. This describes a torus fibered over a circle base (whose base is in the x direction) with complex structure $\tau = Mx + i$ and kahler modulus $\rho = i$. As the fiber traverses the base $x \to x+1$ and subsequently $\tau \to \tau + M$. The identification (6.2) is needed to glue the torus with complex structure $\tau + 1$ back to the the torus with complex structure τ. Let's now try to impose the boundary conditions (6.2) through orbifolding. Blindly following the methods developed in this paper the partition function would contain pieces of the form

$$\sum_{a,b} \mathcal{Z}_{R_x}{}^{a}{}_{b}\ \mathcal{Z}_{R(y)_z}{}^{a}{}_{b}\,. \tag{6.3}$$

Here $\mathcal{Z}_{R_x}{}^{a}{}_{b}$ are again the shift orbifold partition traces (3.22). $\mathcal{Z}_{R(y)_z}{}^{a}{}_{b}$ should be something similar, however we should be careful. We want to shift z by a distance y. If y was equal to some rational number we could in principle always do this (although it could involve orbifolds whose order approached infinity). However, y is a continuous variable and for irrational y this can not even be done in principle since it would necessarily involve an infinite order orbifold. Moreover, it is unclear how to properly combine (6.3) with the partition function coming from the y embedding coordinate. Compare this to integrating a rotation: there the rotation operator which is being integrated with a shift is not dependent on any of the coordinates of the fiber or base and all of these complications are avoided.

It is not surprising that this is difficult. Indeed, were we able to form this orbifold theory, with a position dependent orbifold action, we would be able to describe strings propagating on curved backgrounds!

Before concluding we would like to return to an observation made in section 4. To review: There is a phase ambiguity which arises when splitting up a rotation into a left and right handed chiral rotations. As a result, the chiral reflection, for example, is 4th order rather than 2nd order. Moreover, using the results of [26] we see that the partition traces which arise are not modular covariant. However, if one properly groups these partition traces (as in (4.16)) one obtains sectors which appear to arise from a modular covariant 2nd order theory (4.17-4.22). This is suggestive and leads us to make the following proposal. When constructing truly asymmetric orbifolds one should should always sum over spacetime equivalent sectors in order to get a modular covariant theory. In the chiral reflection example this means that the untwisted sector without insertions should include the 4 separate partition traces which arise from twisting or inserting with the identity or a 360 degree chiral shift. As shown above, this suggestion certainly works and has good physical motivation for the chiral reflection. However, we have only worked through a single example. One still needs to ask how generic a feature it is that truly asymmetric orbifolds, which are modular invariant but not modular covariant, maybe resummed into lower order modular covariant partition traces. Perhaps more importantly, we should ask if there is a way of developing a formalism which uses this perspective to avoid the difficulties which arise in the standard approach to asymmetric orbifolds. The authors of this paper a currently investigating these issues.

A. Modular covariance of the shift orbifold

Since modular covariance is a stronger condition than modular invariance it is worthwhile check explicitly that the elements of the shift orbifold (3.22) are indeed modular covariant. In addition to checking modular covariance, this section offers a more explicit description of these partition function elements.

First we will define,

$$\Omega[\alpha,\beta;\gamma,\eta] \equiv \sum_{n,w} \exp\Big[-\pi\tau_2\big((\alpha n+\beta)^2+(\gamma w+\eta)^2\big)+2\pi i\tau_1(\alpha n+\beta)(\gamma w+\eta)\Big] \,. \quad \text{(A.1)}$$

Note that $\Omega[1,0;1,0]$ is proportional to the circle partition function (3.17) at self dual radius, $\mathcal{Z}_\circ$. Making contact with our earlier notation,

$$\Omega[\alpha,\beta;\gamma,\eta] = |\eta(\tau)|^2\mathcal{Z}_\circ[\alpha n+\beta|\gamma w+\eta] \quad \text{(A.2)}$$

Taking $[\alpha,\beta;\gamma,\eta]=[N,q;1,a/N]$ as in (3.22) it is trivial to show that $\mathcal{Z}_{\frac{\circ}{N}}{}^{a}{}_{b}$ is properly behaved under modular shifts, $\tau\to\tau+1$. Moving on to modular inversions, $\tau\to -1/\tau$, its straight forward to show $\Omega[\alpha,\beta;\gamma,\eta]$ becomes $\widetilde{\Omega}[\alpha,\beta;\gamma,\eta]$, where,

$$\widetilde{\Omega}\left[\alpha,\beta;\gamma,\eta\right] \equiv |\tau|\frac{e^{-2\pi i\eta\beta/\tau_1}}{\alpha\gamma}\sum_{p,m}\exp\left[-\pi\tau_2\left(\frac{p^2}{\gamma^2}+\frac{m^2}{\alpha^2}\right)+2\pi i\tau_1\left(\frac{m}{\alpha}+\frac{\eta}{\tau_1}\right)\left(\frac{p}{\gamma}+\frac{\beta}{\tau_1}\right)\right]. \quad \text{(A.3)}$$

Now consider the partition trace $\mathcal{Z}_\circ[Nn+\alpha|w+a/N] = \Omega[N,\alpha;1,a/N]/|\eta(\tau)|^2$. Using (A.3) we find that under $\tau \to -1/\tau$ this becomes,

$$|\tau| \sum_{p,m} \left[\frac{e^{2\pi i m\alpha/N}}{N}\right] e^{2\pi i a p/N} \exp\left[-\pi\tau_2\left(p^2+\frac{m^2}{N^2}\right)+2\pi i\tau_1 p(\frac{m}{N})\right] \tag{A.4}$$

It then follows from (3.22) and (A.4) that,

$$\begin{aligned}
\mathcal{Z}_{\frac{\circ}{N}}(-1/\tau)^a{}_b &= \sum_{p,m} \mathbf{P}[\mathbf{b}]\ e^{2\pi i a p/N} \exp\left[-\pi\tau_2\left(p^2+\frac{m^2}{N^2}\right)+2\pi i\tau_1 p\left(\frac{m}{N}\right)\right] && \text{(A.5)}\\
&= \sum_{p,\tilde{m}} e^{2\pi i a p/N} \exp\left[-\pi\tau_2\left(p^2+\left(\tilde{m}-\frac{b}{N}\right)^2\right)+2\pi i\tau_1 p\left(\tilde{m}+\frac{b}{N}\right)\right] && \text{(A.6)}\\
&= \mathcal{Z}_{\frac{\circ}{N}}(\tau)^{-b}{}_a\,. && \text{(A.7)}
\end{aligned}$$

Here we have used the fact that

$$\mathbf{P}[\mathbf{b}] \equiv \sum_{\alpha=0}^{N-1}\left[\frac{e^{2\pi i\alpha(m+1)/N}}{N}\right] \tag{A.8}$$

is a projector taking $m \to \tilde{m} = 4m - b$. Here we have worked at the self-dual radius to avoid clutter but by rescaling α and γ it is trivial to extend this proof for all radii.

B. Fermionic partition traces

In this appendix we will discuss the contribution to orbifold partition traces coming from fermions. Since this is a key part of the construction in section 4 we felt that it was important to discuss some of the subtleties which arise. Ultimately we will be interested in orbifolds of the heterotic string. We will however simultaneously discuss the type II orbifolds due to there similarities. Namely, in the type II case the fermionic partition trace takes the form [32]

$$Z_\psi^\pm = \frac{1}{2}\left[Z_0^0(\tau)^4 - Z_1^0(\tau)^4 - Z_0^1(\tau)^4 \mp Z_1^1(\tau)^4\right], \tag{B.1}$$

and in the heterotic string the internal directions lead to the partition trace,

$$Z_{16} = \frac{1}{2}\left[Z_0^0(\tau)^{16} + Z_1^0(\tau)^{16} + Z_0^1(\tau)^{16} + Z_1^1(\tau)^{16}\right], \tag{B.2}$$

The exact definition of Z_β^α is given below (B.18). The important point for now is to note the similarities between the traces. It follows that going from the heterotic orbifold to the type II orbifold simply requires changing some signs and powers.

Consider fermionic orbifolds under the action,

$$\psi(z)^a \to e^{2\pi i\lambda}\psi(a)^a\,. \tag{B.3}$$

Here ψ^a is a complex fermion and $\lambda \in \mathbb{Z}/2$. The action appears at first glance to be second order, but we must be careful since we are dealing with fermions. In particular, twisting by (B.3) twice generates a spectral flow. Namely the ground state before the double twist becomes and excited state in the doubly-twisted theory.

After a λ twist[15] the fermionic part of the stress energy tensor is,

$$
\begin{aligned}
L_0^\psi &= \frac{1}{2}\sum_{n\in\mathbb{Z}}(n+\lambda) :\bar\psi^a_{-n-\lambda}\psi^a_{n+\lambda}: && \text{(B.4)}\\
&= \frac{1}{2}\sum_{n\geqslant 0}(n+\lambda)\bar\psi^a_{-n-\lambda}\psi^a_{n+\lambda} - \frac{1}{2}\sum_{n<0}(n+\lambda)\psi^a_{n+\lambda}\bar\psi^a_{-n-\lambda} + a^\psi && \text{(B.5)}\\
&= \frac{1}{2}\sum_{n\geqslant 0}(n+\lambda)\bar\psi^a_{-n-\lambda}\psi^a_{n+\lambda} + \frac{1}{2}\sum_{n\geqslant 0}(n+1-\lambda)\psi^a_{-n-1+\lambda}\bar\psi^a_{n+1-\lambda} + a^\psi && \text{(B.6)}\\
&\equiv \bar l_0 + l_0 + a^\psi && \text{(B.7)}
\end{aligned}
$$

Here a^ψ is the zero p.t. energy. The groundstates of the sector twisted by λ is given by,

$$\psi_\lambda|0\rangle_\lambda = \bar\psi_{1-\lambda}|0\rangle_\lambda = 0 \ . \tag{B.8}$$

Its important to ask how the reflection insertions act on these ground states. Define $\mathcal{R}$ to be a 180 degree rotation in the untwisted theory. It follows that

$$
\begin{aligned}
&|0\rangle_0 \equiv |+\rangle \Rightarrow \mathcal{R}^\beta|+\rangle = i^\beta|+\rangle && \text{(B.9)}\\
&|0\rangle_{1/2} \equiv |0;\mathrm{NS}\rangle \Rightarrow \mathcal{R}^\beta|0;\mathrm{NS}\rangle = |0;\mathrm{NS}\rangle && \text{(B.10)}\\
&|0\rangle_1 \equiv |-\rangle = \bar\psi_0|+\rangle \Rightarrow \mathcal{R}^\beta|-\rangle = i^{-\beta}|-\rangle && \text{(B.11)}\\
&|0\rangle_{3/2} = \bar\psi_{-1/2}|0;\mathrm{NS}\rangle \Rightarrow \mathcal{R}^\beta\bar\psi_{-1/2}|0;\mathrm{NS}\rangle \\
&\qquad = (-)^\beta\bar\psi_{-1/2}\mathcal{R}^\beta|0;\mathrm{NS}\rangle = (-)^\beta\bar\psi_{-1/2}|0;\mathrm{NS}\rangle && \text{(B.12)}
\end{aligned}
$$

Defining $\alpha = 1-2\lambda$, we can see

$$\mathcal{R}^\beta|0\rangle_\alpha = e^{\pi i\alpha\beta/2}|0\rangle_\alpha \tag{B.13}$$

Writing down the sum over all λ-twisted states,

$$|\lambda\rangle \equiv \prod_{m=1}\sum_{F_m,\bar F_m=0}^{1}\psi^{b\ F_m}_{-m+\lambda}\bar\psi^{b\ \bar F_m}_{-m-\lambda+1}|0\rangle_\lambda\,, \tag{B.14}$$

(here b is being summed from 1 to 4) and noting that,

$$l_0|\lambda\rangle = \prod_{m=1}\sum_{F_m,\bar F_m=0}^{1}(m-\lambda)F_m\psi^{b\ F_m}_{-m+\lambda}\bar\psi^{b\ \bar F_m}_{-m-\lambda+1}|0\rangle_\lambda \tag{B.15}$$

$$\bar l_0|\lambda\rangle = \prod_{m=1}\sum_{F_m,\bar F_m=0}^{1}(m+\lambda-1)\bar F_m\psi^{b\ F_m}_{-m+\lambda}\bar\psi^{b\ \bar F_m}_{-m-\lambda+1}|0\rangle_\lambda \tag{B.16}$$

$$\mathcal{R}^\beta|\lambda\rangle = \prod_{m=1}\sum_{F_m,\bar F_m=0}^{1}(-)^{\beta(F_m+\bar F_m)}e^{\pi i\alpha\beta/2}\psi^{b\ F_m}_{-m+\lambda}\bar\psi^{b\ \bar F_m}_{-m-\lambda+1}|0\rangle_\lambda\,, \tag{B.17}$$

[15]Note that taking $\lambda = 0$ gives the untwisted trace as well.

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it is clear that,

$$\mathrm{Tr}_\alpha[q^{L_0^\psi}\mathcal{R}^\beta] \equiv Z^\alpha_\beta = q^{(3\alpha^2-1)/24}e^{\pi i\alpha\beta/2} \times \prod_{m=1}^{\infty}\left[1+e^{\pi i\beta}q^{m-(1-\alpha)/2}\right]\left[1+e^{-\pi i\beta}q^{m-(1+\alpha)/2}\right] \tag{B.18}$$

$$= \frac{1}{\eta(\tau)}\vartheta_{\alpha\beta}(\tau)\,. \tag{B.19}$$

Its worth noting that since we are twisting by $\alpha = 1-2\lambda$, but we are inserting $\mathcal{R}^\beta$, rather than $\mathcal{R}^{1-2\beta}$, we have effectively redefined what we mean by an untwisted fermion. In particular, $\alpha = 0$ corresponds to the untwisted NS sector.

These are not, however, valid partition traces for the $\alpha = 2, 3$ twisted sectors! In deriving (B.18) we inserted the same rotation operator into each sector. Just as the operators ψ_r's (and consequently L_0 and so forth), themselves get twisted, any operator you insert in a twisted sector must also be twisted. We'll explicitly construct this operator below, but first lets try to clarify the main idea. Just as the groundstates depend on "λ" so does the reflection operator, $\mathcal{R} \to \mathcal{R}_\lambda$. Twisting twice takes $|0\rangle_\lambda, \mathcal{R}_\lambda \to |0\rangle_{\lambda+1}, \mathcal{R}_{\lambda+1}$, where action of $\mathcal{R}_{\lambda+1}$ on $|0\rangle_{\lambda+1}$ is identical to the action of $\mathcal{R}_\lambda$ on $|0\rangle_\lambda$. For example

$$\mathcal{R}_{1/2}|0\rangle_{1/2+1} = \mathcal{R}_{1/2}\bar{\psi}_{-1/2}|0\rangle_{1/2} = -|0\rangle_{1/2+1} \tag{B.20}$$

However, when putting a $\mathcal{R}_\lambda$-insertion in our partition function we need to use twisted operator, $\mathcal{R}_{1/2+1}$,

$$\mathcal{R}_{1/2+1}|0\rangle_{1/2+1} = +|0\rangle_{1/2+1} \tag{B.21}$$

We will now construct $\mathcal{R}_\lambda$ explicitly and check that it does indeed behave in this manner. $\mathcal{R}_\lambda$ can easily be expressed in terms of fermion number, however, there is one subtlety. As mentioned above, we are in essence discussing spectral flow (which includes intermediate values of λ). It follows that the fermion number operator we write down *must* be able to smoothly vary with λ. For this reason we choose to work with the following fermion number operator,

$$\mathcal{F}_\lambda = \sum_{r\in Z+\lambda} :\bar{\psi}^a_{-r}\psi^a_r: = \sum_{n\geqslant 0}\bar{\psi}^a_{-n-\lambda}\psi^a_{n+\lambda} - \sum_{n\geqslant 0}\psi^a_{-n-1+\lambda}\bar{\psi}^a_{n+1-\lambda}\,. \tag{B.22}$$

Its easy to check that the rotation operator,

$$\mathcal{R}_\lambda = e^{\pi i\mathcal{F}_\lambda}\ , \tag{B.23}$$

gives a rotation by π when acting on worldsheet fermions.

Now consider taking $\lambda \to \lambda+1$. The new ground state, $|0\rangle_{\lambda+1}$, satisfies,

$$\psi_{n+\lambda+1}|0\rangle_{\lambda+1} = \bar{\psi}_{n-\lambda}|0\rangle_{\lambda+1} = 0 \ \ ; \quad n \geqslant 0\ , \tag{B.24}$$

so that the operator,

$$\mathcal{F}_{\lambda+1} = \sum_{n\geqslant 0}\bar{\psi}^a_{-n-1-\lambda}\psi^a_{n+1+\lambda} - \sum_{n\geqslant 0}\psi^a_{-n+\lambda}\bar{\psi}^a_{n-\lambda}\ , \tag{B.25}$$

is still normal ordered (with respect to the new vacuum) and subsequently behaves exactly the same way as $\mathcal{F}_\lambda$ behaved on $|0\rangle_\lambda$. Note however the operator $\mathcal{F}_\lambda$ is not normal ordered with respect to the new vacuum since the, $\bar{\psi}_{-\lambda}\psi_\lambda$ term is not normal ordered. Commuting these operators we can see that $\mathcal{F}_\lambda = \mathcal{F}_{\lambda+1} + 1$ or that

$$\mathcal{R}_{\lambda+1} = e^{\pi i}\mathcal{R}_\lambda \tag{B.26}$$

Lets now return to our expression (B.18). It should now be clear that for the $\alpha = 0, 1$ sectors (B.18) is correct, this is correct but for the $\alpha = 2, 3$ sectors the terms with $\beta = 1, 3$ are off by a minus sign. The correct α, β sectors for our orbifold are then,

$$\mathcal{P}^\alpha_\beta = (-)^{\alpha(\alpha-1)\beta/2} Z^\alpha_\beta \tag{B.27}$$

Note that this has the properties,

$$\mathcal{P}^{\alpha+2}_\beta = (-)^\beta \mathcal{P}^\alpha_\beta \quad , \quad \mathcal{P}^\alpha_{\beta+2} = (-)^\alpha \mathcal{P}^\alpha_\beta \tag{B.28}$$

It follows that if we start with α, β-sector, $Z^{\alpha\ N}_\beta$, and do a/b twists/insertions on n of the dimensions you find,

$$Z^{\alpha\ N}_\beta \to Z^{\alpha\ N-n}_\beta \mathcal{P}^{\alpha-a\ n}_{\beta+b} \tag{B.29}$$

The orbifolded fermion partition function is then $\sum_{a,b=0}^3 \lambda^a_b$. Where,

$$\text{(HET)} \qquad \lambda^a_b \equiv \sum_{\alpha,\beta=0}^{1} \mathrm{sgn}_{ab}[\alpha,\beta]^n Z^{\alpha\ 16-n}_\beta Z^{||\alpha-a||\ n}_{||\beta+b||} \tag{B.30}$$

$$\text{(TypeII)} \qquad \lambda^a_b \equiv \sum_{\alpha,\beta=0}^{1} (-)^{\alpha\beta+\alpha+\beta}(\mathrm{sgn}_{ab}[\alpha,\beta])^n Z^{\alpha\ 4-n}_\beta Z^{||\alpha-a||\ n}_{||\beta+b||} \ . \tag{B.31}$$

Here $||m|| \equiv (1-(-)^m)/2$, *i.e.* 0 if m is even and 1 if m is odd. The values for $\mathrm{sgn}_{ab}[\alpha,\beta]$ can be found using (B.28) and the transformation properties of (B.19), and have been given in Table 1. The factor of $(-)^{\alpha\beta+\alpha+\beta}$ in the type II case comes from the relative signs in (B.1).

C. Gradient energy, monodromies and moduli

Let us now show why the fields ρ and τ must be fixed under the action of the monodromy. It follows from (3.1) that if one allows the moduli of the fiber to vary there is a nontrivial backreaction on the base. Such a backreaction is not possible on a T^2. Let us be more explicit. The metric on the base in complex coordinates takes the form

$$ds^2_{base} = \frac{\tilde{\rho}_2(z,\bar{z})}{\tilde{\tau}_2}|\tilde{\tau}d\tilde{\theta}_1 + d\tilde{\theta}_2|^2 \equiv \frac{\tilde{\rho}_2(z,\bar{z})}{\tilde{\tau}_2}|dz|^2 \, , \tag{C.1}$$

this is just the usual metric written in terms of the moduli $\tilde{\tau}$ and $\tilde{\rho}$. It follows that Ricci scalar R is a total derivative:

$$R = -\nabla^2_{base} \ln \tilde{\rho}_2 = -\frac{\tilde{\tau}_2}{\tilde{\rho}_2}\partial\bar{\partial}\ln\tilde{\rho}_2 \tag{C.2}$$

$$= -\frac{\tilde{\tau}_2}{\tilde{\rho}_2}\partial\bar{\partial}\ln\rho_2\tau_2 = -\nabla^2_{base}\ln\rho_2\tau_2 \ . \tag{C.3}$$

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Going from the first line to the second line of (C.2) we have used (3.1). The requirement that the base be a T^2 forces the Euler characteristic to vanish;

$$\chi_{base} = \int_{T^2_{base}} R = 0\,. \tag{C.4}$$

Since R is a total derivative this is trivially satisfied in compactifications where ρ_2 and τ_2 are single valued. However, except in the trivial case where ρ and τ are constants and take values at fixed points of the monodromy, if τ_2 or ρ_2 undergoes a nontrivial monodromy the surface term does not vanish. We can now see that in order to satisfy (C.4) and have a nontrivial monodromy, ρ_2 and τ_2 must be fixed under the action of the monodromy.

D. ζ-functions in two sentences

The appendices of [26] review in detail ϑ and ζ functions. Since the ζ functions are less familiar to most readers we define them, and give some of their properties below. The reader is encouraged to also see the appendix of [26] which gives additional properties, as well as specific modular transformations of the ζ functions which we use in this paper.

The ζ functions are a two parameter family of functions defined by

$$\zeta_{\alpha\beta}(\tau) \equiv e^{-2\pi i\alpha\beta}\frac{\vartheta_{\alpha\beta}(0,2\tau)}{\eta(\tau)} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{(n+\alpha)^2}e^{2\pi i n\beta}\,. \tag{D.1}$$

The periodicity properties of (D.1) are

$$\begin{aligned} \zeta_{(-\alpha)(-\beta)} &= \zeta_{\alpha\beta} \\ \zeta_{(\alpha+1)\beta} &= e^{-2\pi i\beta}\zeta_{\alpha\beta} \\ \zeta_{\alpha(\beta+1)} &= \zeta_{\alpha\beta}, \end{aligned} \tag{D.2}$$

and under modular transformations

$$\begin{aligned} \zeta_{\alpha\beta}(\tau+1) &= e^{-i\pi/12}e^{2\pi i\alpha^2}\zeta_{\alpha(\beta+2\alpha)}(\tau) \\ \zeta_{\alpha\beta}(-\frac{1}{\tau}) &= \frac{1}{\sqrt{2}}e^{-2\pi i\alpha\beta}\left(\zeta_{(-\frac{\beta}{2})2\alpha}(\tau) + e^{2\pi i\alpha}\zeta_{(\frac{1}{2}-\frac{\beta}{2})2\alpha}(\tau)\right)\,. \end{aligned} \tag{D.3}$$

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